Spin-structures on real Bott manifolds

A. Gąsior*

1 Introduction

Let M^n be a flat manifold of dimension n, i.e. a compact connected Riemannian manifold without boundary with zero sectional curvature. From the theorem of Bieberbach ([2], [16]) the fundamental group $\pi_1(M^n) = \Gamma$ determines a short exact sequence:

$$0 \to \mathbb{Z}^n \to \Gamma \stackrel{p}{\to} G \to 0, \tag{1}$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and G is a finite group which is isomorphic to the holonomy group of M^n. The universal covering of M^n is the Euclidean space $\mathbb{R}^n$ and hence Γ is isomorphic to a discrete cocompact subgroup of the isometry group $\mathrm{Isom}(\mathbb{R}^n) = \mathrm{O}(n) \times \mathbb{R}^n = E(n)$. In that case $p : \Gamma \to G$ is a projection on the first component of the semidirect product $O(n) \ltimes \mathbb{R}^n$ and $\pi_1(M_n) = \Gamma$ is a subfroup of $O(n) \ltimes \mathbb{R}^n$. Conversely, given a short exact sequence of the form (1), it is known that the group Γ is (isomorphic to) the fundamental group of a flat manifold if and only if Γ is torsion free. In this case Γ is called a Bieberbach group. We can define a holonomy representation $\phi : G \to \mathrm{GL}(n, \mathbb{Z})$ by the formula:

$$\forall e \in \mathbb{Z}^n \; \forall g \in G, \phi(g)(e) = \tilde{g}e(\tilde{g})^{-1}, \tag{2}$$

where $p(\tilde{g}) = g$. In this article we shall consider Bieberbach groups of rank n with holonomy group $\mathbb{Z}_2^k$, $1 \leq k \leq n-1$, and $\phi(\mathbb{Z}_2^k) \subset D \subset \mathrm{GL}(n, \mathbb{Z})$. Here D is the group of matrices with ± 1 on the diagonal.

Let

$$M_n \stackrel{\mathbb{R}P^1}{\to} M_{n-1} \stackrel{\mathbb{R}P^1}{\to} \ldots \stackrel{\mathbb{R}P^1}{\to} M_1 \stackrel{\mathbb{R}P^1}{\to} M_0 = \{\bullet\} \tag{3}$$

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be a sequence of real projective bundles such that $M_i \to M_{i-1}$, $i = 1, 2, \ldots, n$, is a projective bundle of a Whitney sum of a real line bundle L_{i-1} and the trivial line bundle over M_{i-1}. The sequence (3) is called the real Bott tower and the top manifold M_n is called the real Bott manifold, [3].

Let γ_i be the canonical line bundle over M_i and we set $x_i = w_1(\gamma_i)$ (w_1 is the first Stiefel-Whitney class). Since $H^1(M_{i-1}, \mathbb{Z}_2)$ is additively generated by $x_1, x_2, .., x_{i-1}$ and L_{i-1} is a line bundle over M_{i-1}, we can uniquely write

$$w_1(L_{i-1}) = \sum_{k=1}^{i-1} a_{ki} x_k \tag{4}$$

where $a_{ki} \in \mathbb{Z}_2$ and $i = 2, 3, ..., n$.

From above we obtain the matrix $A = [a_{ki}]$ which is a $n \times n$ strictly upper triangular matrix whose diagonal entries are 0 and remaining entries are either 0 or 1. One can observe (see [11]) that the tower (3) is completly determined by the matrix A and therefore we may denote the real Bott manifold M_n by $M(A)$. From [11, Lemma 3.1] we can consider $M(A)$ as the orbit space $M(A) = \mathbb{R}^n/\Gamma(A)$, where $\Gamma(A) \subset E(n)$ is generated by elements

$$s_i = \left(\begin{bmatrix} 1 & 0 & 0 & . & . & \ldots & 0 \\ 0 & 1 & 0 & . & . & \ldots & 0 \\ . & . & . & . & . & \ldots & \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & (-1)^{a_{i,i+1}} & \ldots & 0 \\ . & . & . & . & . & \ldots & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & (-1)^{a_{i,n}} \end{bmatrix}, \begin{pmatrix} 0 \\ . \\ 0 \\ \frac{1}{2} \\ 0 \\ . \\ 0 \\ 0 \end{pmatrix} \right) \in E(n), \tag{5}$$

where $(-1)^{a_{i,i+1}}$ is in the $(i+1, i+1)$ position and $\frac{1}{2}$ is the $i-$th coordinate of the column, $i = 1, 2, ..., n-1$. $s_n = \left(I, \left(0, 0, ..., 0, \frac{1}{2}\right)\right) \in E(n)$. From [11, Lemma 3.2, 3.3] $s_1^2, s_2^2, ..., s_n^2$ commute with each other and generate a free abelian subgroup $\mathbb{Z}^n$. In other words $M(A)$ is a flat manifold with holonomy group Z_2^k of diagonal type. Here k is a number of non zero rows of a matrix A.

We have the following two lemmas.

Lemma 1.1 ([11], Lemma 2.1)**.** *The cohomology ring $H^*(M(A), \mathbb{Z}_2)$ is generated by degree one elements $x_1, \ldots, x_n$ as a graded ring with n relations*

$$x_j^2 = x_j \sum_{i=1}^{n} a_{ij} x_i,$$

for $j = 1, \ldots, n$.

Lemma 1.2 ([11], Lemma 2.2). *The real Bott manifold $M(A)$ is orientable if and only if the sum of entries is* $0 (\mathrm{mod}\, 2)$ *for each row of the matrix A.*

There are a few ways to decide whether there exists a Spin-structure on an oriented flat manifold M^n. We start with

Definition 1.1 ([5]). An oriented flat manifold M^n has a Spin-structure if and only if there exists a homomorphism $\epsilon\colon \Gamma \to \mathrm{Spin}(n)$ such that $\lambda_n \epsilon = p$, where $\lambda_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the covering map.

There is an equivalent condition for existence of Spin-structure. This is well known ([5]) that the closed oriented differential manifold M has a Spin-structure if and only if the second Stiefel-Whitney class vanishes.

The k-th Stiefel-Whitney class [12, page 3, (2.1)] is given by the formula

$$w_k(M(A)) = (B(p))^* \sigma_k(y_1, y_2, ..., y_n) \in H^k(M(A); \mathbb{Z}_2), \tag{6}$$

where σ_k is the k-th elementary symmetric function, $B(p)$ is a map induced by p on the classification space and

$$y_i := w_1(L_{i-1}) \tag{7}$$

for $i = 2, 3, \ldots, n$. Hence,

$$w_2(M(A)) = \sum_{1<i<j\leq n} y_i y_j \in H^2(M(A); \mathbb{Z}_2). \tag{8}$$

Definition 1.2. ([3], page 4) A binary square matrix A is a Bott matrix if $A = PBP^{-1}$ for a permutation matrix P and a strictly upper triangular binary matrix B.

Our paper is a sequel of [8]. There are given some conditions of the existence of Spin-structures.

Theorem 1.1. ([8], page 1021) *Let A be a matrix of an orientable real Bott manifold $M(A)$.*

1. *Let $l \in \mathbb{N}$ be an odd number. If there exist $1 \leq i < j \leq n$ and rows $A_{i,*}$, $A_{j,*}$ such that*

   $$\sharp\{m : a_{i,m} = a_{j,m} = 1\} = l$$

   and

   $$a_{ij} = 0$$

   then $M(A)$ has no Spin-structure.

2. If $a_{ij} = 1$ and there exist $1 \le i < j \le n$ and rows

$$A_{i,*} = (0, \dots, 0, a_{i,i_1}, \dots, a_{i,i_{2k}}, 0, \dots, 0),$$
$$A_{j,*} = (0, \dots, 0, a_{j,i_{2k+1}}, \dots, a_{j,i_{2k+2l}}, 0, \dots, 0)$$

such that $a_{i,i_1} = \ldots = a_{i,i_{2k}} = 1$, $a_{i,m} = 0$ for $m \notin \{i_1, \dots, i_{2k}\}$, $a_{j,i_{2k+1}} = \ldots = a_{j,i_{2k+2l}} = 1$, $a_{j,r} = 0$ for $r \notin \{i_{2k+1}, \dots, i_{2k+2l}\}$ and l, k are odd then $M(A)$ has no Spin-structure.

In this paper we extend this theorem and we formulate necessary and sufficient conditions of the existence of a Spin-structure on real Bott manifolds. Here is our main result for Bott manifolds with holonomy group Z_2^k, k even. Here is our main result

Theorem 1.2. *Let A be a Bott matrix with k non zero rows where k is an even number. Then the real Bott manifold manifold $M(A)$ has a Spin-structure if and only if for all $1 \le i < j \le n$ manifolds $M(A_{ij})$ have a Spin-structure, where A_{ij} is a matrix with $i-$ and $j-$ th nonzero rows.*

The structure of a paper is as follows. In Section 2 we give three lemmas. First of them gives a decomposition of the $n \times n-$integer matrix A into $n \times n-$integer matrices A_{ij} with $i-$th and $j-$th nonzero rows. In Lemmas 2.2. and 2.3 we examine dependence of y_i and w_2 of a real Bott manifold $M(A)$ on values y_i^{jk} and $w_2(M(A_{jk}))$ of manifolds $M(A_{jk})$. Then the proof of Theorem 1.2 will follow from Lemmas 2.2. and 2.3. Section 3 has a very technical character. In this section we shall give a complete characterization of the existence of the Spin-structure on manifolds $M(A_{ij})$, $1 \le i < j \le n$. Almost all statements in part 2 and 3 are illustrated by examples.

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2 Proof of the Main Theorem

At the beginning we give formula for the decomposition of real Bott matrix A into the sum of the real Bott matrices with two nonzero rows.

Lemma 2.1. *Let A be $n \times n-$Bott matrix and let A_{ij}, $1 \le i < j \le n$, be $n \times n$-matrices with $i-$th and $j-$th nonzero rows. Then, if k is even, we have the following decomposition*

$$A = \sum_{1 \le i < j \le n} A_{ij}. \tag{9}$$

Proof. Let A be $n \times n$-Bott matrix with k nonzero rows, k is an even number. Without loss of generality we can assume that nonzero rows have numbers from 1 to k. We shall consider the matrix A as a sum of matrices A_{ij}, $1 \leq i < j \leq n$. The number of matrices A_{ij} is equal $\binom{k}{2}$. For $1 \leq i \leq k$ there are $(k-1)$-two elements subsets of $\{1, 2, \ldots, k\}$ containing i. Thus having summed matrices A_{ij} we obtain

$$(k-1) \cdot A = \sum_{1 \leq i < j \leq n} A_{ij}. \tag{10}$$

Since A is Bott matrix and k is an even number we get the formula (9).

□

Example 2.1. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $n = 6$, $k = 4$, so we have

$$A = \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{12}} + \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{13}} + \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{14}}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{23}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{24}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{34}}$$

Before we start a proof of the main theorem we give an example.

Example 2.2. For the manifold $M(A)$ from Example 2.1 we get

$$y_2 = x_1, \ y_3 = x_1 + x_2, \ y_4 = x_2 + x_3, \ y_5 = x_3 + x_4, \ y_6 = x_4.$$

Hence
$$\omega_2(M(A)) = x_1x_3 + x_2x_4.$$

We compute second Stiefel-Whitney classes for real Bott manifolds $M(A_{ij})$ from Example 2.1. For these purpose we put $y_l^{ij} = w_1(L_{l-1})$ for manifolds $M(A_{ij})$ and we obtain

$$\begin{array}{llllll}
y_2^{12} = x_1 & y_2^{13} = x_1 & y_2^{14} = x_1 & y_2^{23} = 0 & y_2^{24} = 0 & y_2^{34} = 0 \\
y_3^{12} = x_1 + x_2 & y_3^{13} = x_1 & y_3^{14} = x_1 & y_3^{23} = x_2 & y_3^{24} = x_2 & y_3^{34} = 0 \\
y_4^{12} = x_2 & y_4^{13} = x_3 & y_4^{14} = 0 & y_4^{23} = x_2 + x_3 & y_4^{24} = x_2 & y_4^{34} = x_3 \\
y_5^{12} = 0 & y_5^{13} = x_3 & y_5^{14} = x_4 & y_5^{23} = x_3 & y_5^{24} = x_4 & y_5^{34} = x_3 + x_4 \\
y_6^{12} = 0 & y_6^{13} = 0 & y_6^{14} = x_4 & y_6^{23} = 0 & y_6^{24} = x_4 & y_6^{34} = x_4
\end{array}$$

With the above notation we get

$$\begin{aligned}
&\sum_{1\le i<j\le k} y_2^{ij} = 3x_1 = x_1 \Rightarrow \sum_{1\le i<j\le k} y_2^{ij} = y_2, \\
&\sum_{1\le i<j\le k} y_3^{ij} = 3x_1 + 3x_2 = x_1 + x_2 \Rightarrow \sum_{1\le i<j\le k} y_3^{ij} = y_3, \\
&\sum_{1\le i<j\le k} y_4^{ij} = 3x_2 + 3x_3 = x_2 + x_3 \Rightarrow \sum_{1\le i<j\le k} y_4^{ij} = y_4, \\
&\sum_{1\le i<j\le k} y_5^{ij} = 3x_3 + 3x_4 = x_3 + x_4 \Rightarrow \sum_{1\le i<j\le k} y_5^{ij} = y_5, \\
&\sum_{1\le i<j\le k} y_6^{ij} = 3x_4 = x_4 \Rightarrow \sum_{1\le i<j\le k} y_6^{ij} = y_6
\end{aligned}$$

and second Stiefel-Whitney classes for manifolds $M(A_{ij})$ are follows

$$\begin{aligned}
w_2(M(A_{12})) &= 0, \\
w_2(M(A_{13})) &= x_1x_3, \\
w_2(M(A_{14})) &= 0, \\
w_2(M(A_{23})) &= 0, \\
w_2(M(A_{24})) &= 0x_2x_4, \\
w_2(M(A_{34})) &= 0.
\end{aligned}$$

Hence
$$\sum_{1=i<j=4} \omega_2(M(A_{ij}) = x_1x_3 + x_2x_4 = \omega_2(M(A)).$$

Following the method described in the above example we have lemmas.

Lemma 2.2. *Let A be a $n \times n$ Bott matrix with $k > 3$ nonzero rows, k is an even number. Then*

$$y_l = \sum_{1 \le i < j \le k} y_l^{ij}, \tag{11}$$

where $y_l = \omega_1(L_{l-1}(M(A))$ and $y_l^{ij} = \omega_1(L_{l-1}(M(A_{ij}))$.

Proof. We have

$$y_l = w_1(L_{l-1}) = \sum_{k=1}^{l-1} a_{kl} x_k = x \cdot A^l$$

where $x = [x_1, \dots, x_n]$, $A = [a_{ij}]$, A^l is the $l-$th column of the matrix A and $\cdot$ is multiplication of matrices. Let us multiply (9)

on the left by x

$$x \cdot A = \sum_{1 \le i < j \le k} x \cdot A_{ij}.$$

Since $yx \cdot A = [y_1, y_2, \dots, y_n]$ and $x \cdot A^{ij} = [y_1^{ij}, y_2^{ij}, \dots, y_n^{ij}]$, we get (11).

$\square$

Lemma 2.3. *Let A be $n \times n$ Bott matrix with $k-$nonzero rows, $k \ge 4$, k is an even number. Then*

$$w_2(M(A)) = \sum_{1 \le i < j \le k} w_2(M(A_{ij})).$$

Proof. From (8) and (11)

$$\begin{aligned}
\omega_2(M(A)) &= \sum_{l<r} y_l y_r \\
&= \sum_{l<r} \left[\left(\sum_{i<j} y_l^{ij} \right) \right] \left[\left(\sum_{i<j} y_r^{ij} \right) \right] = \sum_{l<r} \left(\sum_{i<j} y_l^{ij} y_r^{ij} \right) \\
&= \sum_{i<j} \left(\sum_{l<r} y_l^{ij} y_r^{ij} \right) = \sum_{i<j} \omega_2(M(A_{ij})).
\end{aligned}$$

$\square$

From proofs of Lemma 2.2 and Lemma 2.3 we obtain a proof of Main Theorem 1.2.

Proof of Theorem 1.2 Let us recall the manifold M has a Spin-structure if and only if $w_2(M) = 0$. At the beginning let us assume, for each pair $1 \le i < j \le n$, we have $w_2(M(A_{ij})) = 0$. Then from Lemma 2.3 we have

$$w_2(M(A)) = \sum_{1 \le i < j \le k} w_2(M(A_{ij})) = 0,$$

so the real Bott manifold $M(A)$ has a Spin-structure.

On the other hand, let the manifold $M(A)$ admits the Spin-structure, then

$$0 = w_2(M(A)) = \sum_{1 \leq i < j \leq k} w_2(M(A_{ij})).$$

Second Stiefel-Whitney classes $M(A_{ij})$ are non negative so

$$\forall_{1 \leq i < j \leq n} w_2(M(A_{ij})) = 0.$$

□

Remark 2.1. *We do not know how to prove the main theorem for odd k. From the other side we are not sure if we can formulate it as a conjecture in this case.*

In the next section of our paper we concentrate on calculations of Spin-structure on manifolds A_{ij}.

3 Existence of Spin-structure on manifolds $M(A_{ij})$

From now, let A be a matrix of an orientable real Bott manifold $M(A)$ of dimension n with two non-zero rows. From Lemma 1.2 we have that the number of entries 1, in each row, is an odd number and we have following three cases:
CASE I. There are no columns with double entries 1,
CASE II. The number of columns with double entries 1 is an odd number,
CASE III. The number of columns with double entries 1 is an even number,

We give conditions for an existence of the Spin-structure on $M(A_{ij})$. In the further part of the paper we adopt the notation $0_p = (\underbrace{0, \ldots, 0}_{p\text{ - times}})$. From the definition, rows of number i and j correspond to generators s_i, s_j which define a finite index abelian subgroup $H \subset \pi_1(M(A))$ (see [9]).

Theorem 3.1. *Let A be a matrix of an orientable real Bott manifold $M(A)$ from the above case I. If there exist $1 \leq i < j \leq n$ such that*
1.

$$\begin{aligned} A_{i,*} &= (0_{i_1}, a_{i,i_1+1}, \ldots, a_{i,i_1+2k}, 0_{i_{2l}}, 0_{i_p}) \\ A_{j,*} &= (0_{i_1}, 0_{i_{2k}}, a_{j,i_1+2k+1}, \ldots, a_{j,i_1+2k+2l}, 0_{i_p}), \end{aligned}$$

where $a_{i,i_1+1} = \ldots = a_{i,i_1+2k} = 1, a_{i,m} = 0$ *for* $m \notin \{i_1, \ldots, i_1 + 2k\}$, $a_{j,i_1+2k+1} = \ldots = a_{j,i_1+2k+2l} = 1, a_{j,r} = 0$ *for* $r \notin \{i_1+2k+1, \ldots, i_1+2k+2l\}$. *Then* $M(A)$ *admits the Spin-structure if and only if either* l *is an even number or* l *is an odd number and* $j \notin \{i_1 + 1, \ldots, i_1 + 2k\}$.
2.

$$\begin{aligned} A_{i,*} &= (0_{i_1}, 0_{i_{2k}}, a_{i,i_{2k}+1}, \ldots, a_{i,i_{2k}+2l}, 0_{i_p}) \\ A_{j,*} &= (0_{i_1}, a_{j,i_1+1}, \ldots, a_{j,i_1+2k}, 0_{i_{2l}}, 0_{i_p}), \end{aligned}$$

where $a_{j,i_1+1} = \ldots = a_{j,i_1+2k} = 1, a_{j,m} = 0$ *for* $m \notin \{i_1, \ldots, i_1+2k\}$, $a_{i,i_{2k}+1} = \ldots = a_{i,i_{2k}+2l} = 1, a_{i,r} = 0$ *for* $r \notin \{i_{2k} + 1, \ldots, i_{2k} + 2l\}$, *then* $M(A)$ *has the Spin-structure.*

Proof. 1. From (7) we have

$$\begin{aligned} y_{i_1+1} = \ldots = y_{i_1+2k} = x_i, \\ y_{i_1+2k+1} = \ldots = y_{i_1+2k+2l} = x_j. \end{aligned}$$

Using (8) and $x_i^2 = x_i \sum_{j=1}^{n} a_{ji}x_j$ we get

$$\begin{aligned} w_2(M(A)) &= k(2k-1)x_i^2 + 4klx_ix_j + l(2l-1)x_j^2 \\ &= k(2k-1)x_i^2 + l(2l-1)x_j^2 = l(2l-1)x_j^2 = lx_j^2. \end{aligned}$$

Summing up, we have to consider the following cases

1. if $l = 2b$, then $w_2(M(A)) = 2bx_j^2 = 0$. Hence $M(A)$ has a Spin-structure,

2. if $l = 2b + 1$, then

$$\begin{aligned} w_2(M(A)) &= (2b+1)x_j^2 = x_j^2 \\ &= \begin{cases} 0, & \text{if } j \notin \{i_1+1, \ldots, i_1+2k\}, M(A) \text{ has a Spin-structure,} \\ x_ix_j, & \text{if } j \in \{i_1+1, \ldots, i_1+2k\}, M(A) \text{ has no Spin-structure.} \end{cases} \end{aligned}$$

2. From (7)

$$\begin{aligned} y_{i_1} + 1 = \ldots = y_{i_1+2k} = x_j \\ y_{i_1+2k+1} = \ldots = y_{i_1+2k+2l} = x_i. \end{aligned}$$

Moreover, from (8) and since $i_1 > j > i$

$$\begin{aligned} w_2(M(A)) &= k(2k-1)x_j^2 + 4klx_ix_j + l(2l-1)x_i^2 \\ &= k(2k-1)\underbrace{x_j^2}_{=0} + l(2l-1)\underbrace{x_i^2}_{=0} = 0. \end{aligned}$$

Hence $M(A)$ has the Spin-structure.

□

Theorem 3.2. *Let A be a matrix of an orientable real Bott manifold $M(A)$ from the above case II. If there exist $1 \le i < j \le n$ such that*
1.

$$\begin{aligned} A_{i,*} &= (0_{i_1}, a_{i,i_1+1}, \dots, a_{i,i_1+2k}, a_{i,i_1+2k+1}, \dots, a_{i,i_1+2k+2l}, 0_{i_{2m}}, 0_{i_p}) \\ A_{j,*} &= (0_{i_1}, 0_{i_{2k}}, a_{j,i_1+2k+1}, \dots, a_{j,i_1+2k+2l}, a_{j,i_1+2k+2l+1}, \dots, a_{j,i_1+2k+2l+2m}, 0_{i_p}), \end{aligned}$$

where $a_{i,i_1+1} = \ldots = a_{i,i_1+2k} = a_{i,i_1+2k+1} = \ldots = a_{i,i_1+2k+2l} = 1, a_{i,r} = 0$ for $r \notin \{i_1+1, \dots, i_1+2k+2l\}$, $a_{j,i_1+2k+1} = \ldots = a_{j,i_1+2k+2l+2m} = 1, a_{j,s} = 0$ for $s \notin \{i_1+2k+1, \dots, i_1+2k+2l+2m\}$.

Then $M(A)$ has the Spin-structure if and only if either l and m are number of the same parity or l and m are number of different parity and $j \notin \{i_1+1, \dots, i_1+2k\}$.
2.

$$\begin{aligned} A_{i,*} &= (0_{i_1}, 0_{i_1+2k}, a_{i,i_1+2k+1}, \dots, a_{i,i_1+2k+2l}, a_{i,i_1+2k+2l+1}, \dots, a_{i,i_1+2k+2l+2m}, 0_{i_p}), \\ A_{j,*} &= (0_{i_1}, a_{j,i_1+1}, \dots, a_{j,i_1+2k}, a_{j,i_1+2k+1}, \dots, a_{j,i_1+2k+2l}, 0_{i_{2m}}, 0_{i_p}) \end{aligned}$$

where $a_{j,i_1+1} = \ldots = a_{j,i_1+2k} = a_{j,i_1+2k+1} = \ldots = a_{j,i_1+2k+2l} = 1, a_{j,m} = 0$ for $m \notin \{i_1+1, \dots, i_1+2k+2l\}$, $a_{i,i_1+2k+1} = \ldots = a_{i,i_1+2k+2l} = a_{i,i_1+2k+2l+1} = \ldots = a_{i,i_1+2k+2l+2m} = 1, a_{i,r} = 0$ for $r \notin \{i_1+2k+1, \dots, i_1+2k+2l+2m\}$, then $M(A)$ has the Spin-structure

Proof. 1. From (7) we have

$$\begin{aligned} y_{i_1+1} = \ldots = y_{i_1+2k} &= x_i, \\ y_{i_1+2k+1} = \ldots = y_{i_1+2k+2l} &= x_i + x_j \\ y_{i_1+2k+2l+1} = \ldots = y_{i_1+2k+2l+2m} &= x_j. \end{aligned}$$

From (8) and $x_i^2 = x_i \sum_{j=1}^{n} a_{ji} x_j$ we get

$$\begin{aligned} w_2(M(A)) &= k(2k-1)x_i^2 + 4klx_i(x_i+x_j) + l(2l-1)(x_i+x_j)^2 + m(2m-1)x_j^2 \\ &= l(2l-1)x_j^2 + m(2m-1)x_j^2 = (l+m)x_j^2. \end{aligned}$$

We have to consider the following cases:

1. If $l+m$ is an even number then $w_2(M(A)) = 0$. Hence $M(A)$ has a Spin-structure.
2. If $l+m$ is an odd number then

$$\begin{aligned} w_2(M(A)) &= x_j^2 \\ &= \begin{cases} 0, & \text{if } j \notin \{i_1+1, \dots, i_1+2k\}, M(A) \text{ has a Spin-structure} \\ x_i x_j, & \text{if } j \in \{i_1+1, \dots, i_1+2k\}, M(A) \text{ has no Spin-structure.} \end{cases} \end{aligned}$$

2. Using (7) we get

$$y_{i_1+1} = \ldots = y_{i_1+1} = x_j$$
$$y_{i_1+2k+1} = \ldots = y_{i_1+2k+2l} = x_i + x_j$$
$$y_{i_1+2k+2l+1} = \ldots = y_{i_1+2k+2l+2m} = x_i.$$

Moreover, from (8) and since $i_1 > j > i$

$$\begin{aligned} & w_2(M(A)) = k(2k-1)x_j^2 + l(2l-1)x_i^2 + 4klx_j(x_i + x_j) + 4kmx_ix_j \\ & + 4lmx_i(x_i + x_j) + l(2l-1)(x_i + x_j)^2 + m(2m-1)x_i^2 \\ & = k(2k-1)\underbrace{x_j^2}_{=0} + l(2l-1)\underbrace{x_i^2}_{=0} + l(2l-1)\underbrace{x_j^2}_{=0} + m(2m-1)\underbrace{x_i^2}_{=0} = 0. \end{aligned}$$

Hence $M(A)$ has a Spin-structure.

□

Theorem 3.3. *Let A be a matrix of an orientable real Bott manifold $M(A)$ from the above case III. If there exist $1 \le i < j \le n$ such that*
1.

$$\begin{aligned} A_{i,*} &= (0_{i_1}, a_{i,i_1+1}, \ldots, a_{i,i_1+2k+1}, a_{i,i_1+2k+2}, \ldots, a_{i,i_1+2k+2l+2}, 0_{i_{2m+1}}, 0_{i_p}) \\ A_{j,*} &= (0_{i_1}, 0_{i_{2k+1}}, a_{j,i_{2k+2}}, \ldots, a_{j,i_1+2k+2l+2}, a_{j,i_1+2k+2l+3}, \ldots, a_{j,i_1+2k+2l+2m+3}, 0_{i_p}), \end{aligned}$$

where $a_{i,i_1+1} = \ldots = a_{i,i_1+2k} = \ldots = a_{i,i_1+2k+2l+2} = 1, a_{i,r} = 0$ for $r \notin \{i_1+1, \ldots, i_1+2k+2l+2\}$, $a_{j,i_1+2k+2} = \ldots = a_{j,i_1+2k+2l+2m+3} = 1, a_{j,s} = 0$ for $s \notin \{i_1+2k+2, \ldots, i_1+2k+2l+2m+3\}$. Then $M(A)$ admits the Spin-structure if and only l and m are number of the same parity and $j \in \{i_1+1, \ldots, i_1+2k+2\}$.
2.

$$\begin{aligned} A_{i,*} &= (0,_{i_1}, 0_{i_{2l+1}}, a_{i,i_1+2k+2}, \ldots, a_{i,i_1+2k+2l+2}, a_{i,i_1+2k+2l+3}, \ldots, a_{i,i_1+2k+2l+2m+3}, 0_{i_p}) \\ A_{j,*} &= (0_{i_1}, a_{j,i_1+1}, \ldots, a_{j,i_1+2k+1}, a_{j,i_1+2k+2}, \ldots, a_{j,i_1+2k+2l+2}, 0_{i_{2m}}, 0_{i_p}) \end{aligned}$$

where $a_{j,i_1+1} = \ldots = a_{j,i_1+2k} = a_{j,i_1+2k+1} = \ldots = a_{j,i_1+2k+2l+2} = 1, a_{j,m} = 0$ for $m \notin \{i_1+1, \ldots, i_1+2k+2l+2\}$, $a_{i,i_1+2k+2} = \ldots = a_{i,i_1+2k+2l+2} = a_{i,i_1+2k+2l+3} = \ldots = a_{i,i_1+2k+2l+2m+3} = 1, a_{i,r} = 0$ for $r \notin \{i_1+2k+2, \ldots, i_1+2k+2l+2m+3\}$. Then $M(A)$ has no Spin-structure.

Proof. 1. From (7)

$$y_{i_1+1} = \ldots = y_{i_1+2k+1} = x_i,$$
$$y_{i_1+2k+2} = \ldots = y_{i_1+2k+2l+2} = x_i + x_j$$
$$y_{i_1+2k+2l+3} = \ldots = y_{i_1+2k+2l+2m+3} = x_j.$$

From (8) and $x_i^2 = x_i \sum_{j=1}^n a_{ji} x_j$ we obtain

$$\begin{aligned} w_2(M(A)) &= k(2k+1)x_i^2 + (2k+1)(2l+1)x_i(x_i+x_j) + (2k+1)(2m+1)x_ix_j \\ &\quad + l(2l+1)(x_i+x_j)^2 + (2l+1)(2m+1)x_j(x_i+x_j) + m(2m+1)x_j^2 \\ &= (l+m+1)x_j^2 + (2l+1)(2m+1)x_ix_j = (l+m+1)x_j^2 + x_ix_j. \end{aligned}$$

Now, if l and m are number of the same parity we have

$$\begin{aligned} &w_2(M(A)) = x_ix_j + x_j^2 \\ &= \begin{cases} x_ix_j, & \text{if } j \notin \{i_1+1, \dots, i_1+2k+2\}, M(A) \text{ has no Spin-structure,} \\ 0, & \text{if } j \in \{i_1+1, \dots, i_1+2k+2\}, M(A) \text{ has a Spin-structure.} \end{cases} \end{aligned}$$

2. From (7)

$$\begin{aligned} y_{i_1+1} &= \ldots = y_{i_1+2k+1} = x_j \\ y_{i_1+2k+2} &= \ldots = y_{i_1+2k+2l+2} = x_i + x_j \\ y_{i_1+2k+2l+3} &= \ldots = y_{i_1+2k+2l+2m+3} = x_i. \end{aligned}$$

From (8) and since $i_1 > j > i$ we get

$$\begin{aligned} w_2(M(A)) &= k(2k+1)x_j^2 + m(2m+1)x_i^2 + (2k+1)(2l+1)x_j(x_i+x_j) \\ &\quad + (2k+1)(2m+1)x_ix_j + l(2l+1)(x_i+x_j)^2 \\ &\quad + (2l+1)(2m+1)x_i(x_i+x_j) + m(2m-1)x_i^2 \\ &= k(2k+1)\underbrace{x_j^2}_{=0} + l(2l+1)\underbrace{(x_i+x_j)^2}_{=0} + m(2m+1)\underbrace{x_i^2}_{=0} \\ &\quad + x_j(x_i+x_j) + x_ix_j + x_i(x_i+x_j) = x_ix_j \neq 0, \end{aligned}$$

so $M(A)$ has no Spin-structure.

$\square$

Now, we give examples which illustrate Theorems 3.1 - 3.3.

Example 3.1. 1. Let

$$A = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Here $2l = 4 \Rightarrow l = 2$. Hence from Theorem 3.1, part 1.1, manifold $M(A)$ has the Spin-structure.

2.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here $l = 1, \{i_1, i_2, \ldots, i_n\} = \{3, 4\}, j = 3 \in \{3, 4\}$. Hence, from Theorem 3.1, part 1.2, the real Bott manifold $M(A)$ has no Spin-structure.

3.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From Theorem 3.2, part 1.4 and since $l = 1, m = 2, \{i_1, \ldots, i_{2k}\} = \{2, 3\}, j = 2 \in \{2, 3\}$ the real Bott manifold has no Spin-structure.

4.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case $l = 1, m = 2$, and from Theorem 3.3 we have that $M(A)$ has no Spin-structure.

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Maria Curie-Skłodowska University,
Institute of Mathematics
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland
E-mail: anna.gasior@poczta.umcs.lublin.pl